J. Fluid Mech. (1971), *vol.* 49, *part* 3, *pp.* 489–507
Printed in Great Britain

Statistical hydromechanics of disperse systems
Part 1. Physical background and general equations

By YU. A. BUYEVICH
Institute for Problems in Mechanics, USSR Academy of Sciences, Moscow, USSR

(Received 6 January 1971)

On the basis of a statistical treatment of the local structure of a flowing monodisperse system of particles suspended in a viscous fluid, a complete set of equations is proposed which governs the mechanical behaviour of both phases looked upon as interpenetrating co-existent continua. The set includes the dynamic equations of mass and momentum conservation of both phases in their mean flow and the kinetic equation for suspended particles. A technique is developed for the calculation of various quantities describing random local motion of the particles and fluid superimposed on their mean flow.

1. Introduction

The motion of various fluid-particle mixtures and transport processes in them are of great interest for many up-to-date applications. Nevertheless, the consistent theory of such mixtures is at present not available despite an enormous number of derivatives of equations for their mean flow. Mostly phenomenological attempts to draw up equations, governing the macroscopic behaviour of disperse systems are known so far (see, for example, Anderson & Jackson 1967, Lype 1965, Murray 1965 and Soo 1967). One common limitation is inherent in most of these endeavours, namely, various terms are introduced into equations empirically on the basis of more or less plausible qualitative hypotheses so that the quantities involved (such as effective viscosities, etc.) are substantially 'things in themselves'. This deficiency, which proves to be unessential only in the case of very dilute mixtures, is perhaps symptomatic of a lack of physical clarity about the properties of a suspension of particles in fluid.

The main difficulty in the construction of a theory of disperse systems is connected with the fact that in their flows the particles and the fluid are in random motion. This motion (it is referred to below as 'pseudo-turbulence') has in many cases a decisive influence upon both rheological properties and transport processes in disperse systems and hence affects their mean flow. For instance, it is this influence that causes the striking increase in the transport coefficients in a fluidized bed as compared with those in a homogeneous fluid. On account of the random character of pseudo-turbulence it is obvious that the theory needed must inevitably be statistical. The necessity for a rigorous statistical treatment of disperse systems has been demonstrated from a somewhat different point of view by Panton (1968) and Tam (1969).

The particles suspended in a viscous fluid represent a complex non-conservative system possessing properties both of a dense gas with potential intermolecular interaction and of a system of interacting Brownian particles in a dissipative medium. The main obstacles to analysis of the system of suspended particles are due just to this fact. There are several papers concerned with the statistical theory of such a system whose authors usually lay emphasis upon one of the two mentioned analogies, clearly at the expense of the other. For example, in Houghton's paper (1966) major attention had been paid to Brownian features of particles and to their interaction with a fluid. On the other hand, Buyevich (1966), Levich & Myasnikov (1966) and Myasnikov (1967) had in fact suggested that interparticle interactions were brought about by means of direct collisions as is the case for a dense gas.

Both methods are obviously insufficient for adequate treatment of the problem and the analogies themselves are rather conventional. First, in contrast to gas molecules, the suspended particles interact not only by means of direct collisions but also through the random fields of the fluid velocity and pressure. Moreover, one might argue about the role of direct collisions altogether. Second, the time scale of the random fluctuation force acting upon some particle coincides with the characteristic time of substantial change in its velocity. Hence it easily follows that properties of suspended particles must differ considerably from those of the Brownian particles. The facts mentioned have stimulated further attempts by the author to construct the improved version of the comprehensive statistical hydromechanics of disperse systems (some references can be found in the papers by Buyevich (1970*a*, *b*)). This paper deals with the formulation of the final version resulting from the investigations mentioned.

The primary aim of the paper is two-fold. First, it is necessary to develop a statistical theory which should give an opportunity to estimate the quantities characterizing random local motion of both phases, pressure and concentration fluctuations, etc. on the microscopic level. Second, equations governing the macroscopic mean flow of both phases looked upon as interpenetrating interacting continua have to be rigorously derived with allowance for the influence of pseudo-turbulence upon this flow. In order to make the treatment more apparent and to leave all the main ideas unencumbered with details, we shall consider below only a monodisperse system whose particles are in a state of thermodynamic equilibrium so that there is no need to take mass and energy exchanges at the interphase boundary into account. Additionally, we shall assume the concept of 'randomness' of pseudo-turbulent motion and shall apply, therefore, the methods used in statistical mechanics. Otherwise one is compelled to consider the complicated many-body problem with small chance of succeeding. We hope that, in spite of a certain complexity of the theory proposed, it gives not only a new interpretation of the physical origin of various properties of two-phase flows, but also a constructive technique for their direct calculation, and will help to provide a more sound foundation for equations of two-phase hydromechanics than that of previous work.

2. Physical discussion and the kinetic equation for suspended particles

We consider the system of uniform particles of equivalent radius a and density d_1 suspended in a fluid whose density and viscosity are d_0 and μ_0 respectively. The situation in the vicinity of each particle is characterized here by the particle velocity $\mathbf{w}$, the local volume concentration ρ, and by the values of the fluid velocity $\mathbf{v}$, and pressure p, averaged over the particle associated volume $\sigma - \sigma_0$ occupied by the fluid, i.e.

$$\left.\begin{aligned} \mathbf{v}(t,\mathbf{r}) &= \frac{1}{\sigma-\sigma_0}\int_\sigma \mathbf{v}^0(t,\mathbf{r}')\,(1-\xi(\mathbf{r}-\mathbf{r}'))\,d\mathbf{r}', \\ p(t,\mathbf{r}) &= \frac{1}{\sigma-\sigma_0}\int_\sigma p^0(t,\mathbf{r}')\,(1-\xi(\mathbf{r}-\mathbf{r}'))\,d\mathbf{r}', \\ \sigma &= \sigma_0/\langle\rho\rangle, \quad \sigma_0 = (\tfrac{4}{3})\pi a^3. \end{aligned}\right\} \tag{2.1}$$

Here the function $\xi(\mathbf{r})$ equals unity inside a particle and zero outside it, σ is the mean specific volume of one particle in the system, $\mathbf{r}$ the radius-vector of the particle centre, $\mathbf{v}^0(\mathbf{r})$ and $p^0(\mathbf{r})$ the true fluid velocity and pressure and $\langle\rho\rangle$ the mean volume concentration of particles in the system. The latter quantity can be, of course, determined by means of the ensemble averaging or through the use of the formula

$$\langle\rho\rangle = \frac{1}{V}\int_V \xi(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}',$$

where integration is performed over some representative volume V which contains a great amount of particles sufficient for the averaging ($V \gg \sigma$) but whose linear size is small as compared with the space scale of the field $\langle\rho\rangle$. The explicit definition of the quantity ρ is not so straightforward as that of the quantities $\mathbf{v}$ and p. The complete theory of the random concentration of disperse systems and of its fluctuations has been given by Buyevich (1970*b*) who used for this purpose some methods of statistical physics and of correlation theory of stationary random processes. The details of the calculation are irrelevant here, and it is sufficient to say that the quantity ρ can be regarded as the ratio of the volume σ_0 to the instantaneous specific volume of the given test particle, the latter being defined in just the same way as in statistical physics of fluids and dense gases. In accordance with this theory, the volume σ is the smallest physical volume of the system under study which can be in principle considered within the limits of any continuum model of disperse systems. It is worth noting that one could readily obtain the generalized definitions (2.1) to introduce $\mathbf{v}$, p as the continuous functions of co-ordinates, if one uses in (2.1) the volumes σ whose centres do not necessarily coincide with those of particles. Note also that these definitions become rather meaningless in the case of a very dilute system when the continuum point of view has no advantage at all and it is reasonable to investigate the motion of discrete particles in a fluid. Really, the volume σ is large in this case so that the accuracy of the continuum description of a dilute system is low.

It is a virtually impossible task to assign a definite position and definite values of ρ, $\mathbf{v}$, p for each particle and to analyze their changes in a direct manner. Therefore, we have to stand on the statistical point of view and regard all these

quantities as random. Let us introduce the distribution function $f(\mathbf{w}; t, \mathbf{r})$ normalized to the mean number concentration of particles, $n(t, \mathbf{r})$. Similarly, the conventional distributions for the quantities ρ, $\mathbf{v}$, p associated with some particle can be introduced, the corresponding value of $\mathbf{w}$ being fixed. This enables us to write the equalities

$$\mathbf{w} = \langle \mathbf{w} \rangle + \mathbf{w}', \quad \rho = \langle \rho \rangle + \rho', \quad \mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}', \quad p = \langle p \rangle + p', \tag{2.2}$$

where the bracket quantities are obtained by averaging over the distributions mentioned and the primed quantities describe the random pseudo-turbulent fluctuations, their mean values being equal to zero. As in other known systems of many particles, it is just the former quantities that appear in the description of the phases as co-existent continua (in a sense, this situation is similar, for example, to that in kinetic theory of gases). These quantities are referred to below as 'dynamic variables'. On the other hand, the primed 'pseudo-turbulent variables' in (2.2) can be treated henceforth as random functions of time and co-ordinates.

The pseudo-turbulent variables define the random motion of both phases of the disperse system under study superimposed upon their mean flow. It may be useful to discuss briefly those physical causes that lead to the generation of this motion. To this end, we consider some volume of the two-phase mixture in the local convective co-ordinate system connected with the mean flow of the particles located in this volume. The origin of this system moves along the laboratory co-ordinate system with the velocity $\langle \mathbf{w} \rangle$, this quantity being assumed uniform over the volume. It is obvious that the mean total force acting upon the particles in the given volume equals zero in the co-ordinate system introduced. This force includes in a general case the force due to the external field and the force of interaction of the particles with the fluid, the latter force being a non-linear function of the local concentration. Therefore, the concentration fluctuations result in violation of the mean force balance so that fluctuating forces arise which accelerate or decelerate individual particles. The energy of the mean flow of the mixture is transferred in this manner to the random motion of individual particles or even of groups of particles. The interparticle interaction brings about a further exchange of energy of these pulsations between all the particles whereas the interphase interaction gives rise to corresponding fluctuations of the fluid velocity and pressure. These interactions add new 'randomness' to the pulsating motion which already is random from the beginning. The energy of this motion is dissipated in the long run by viscous forces but new fluctuations arise at the same time so that one can observe a certain balance between the energy transferred to pseudo-turbulence from the mean flow and the pseudo-turbulent energy dissipated. The existence of random motion of particles and a fluid in two-phase flows is substantiated also by numerous experiments. A more complete discussion of this phenomenon can be found in the Buyevich's paper (1966).

We assume further that the distribution function $f(\mathbf{w}; t, \mathbf{r})$ is governed by the Kolmogoroff–Chapman equation, i.e.

$$f(\mathbf{w}; t, \mathbf{r}) = \iint W(\Delta\mathbf{r}, \Delta\mathbf{w}, \Delta t | \mathbf{r}_0, \mathbf{w}_0, t_0) f(\mathbf{w}_0; t_0, \mathbf{r}_0)\, d\mathbf{w}_0 d\mathbf{r}_0, \tag{2.3}$$

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0, \quad \Delta\mathbf{w} = \mathbf{w} - \mathbf{w}_0, \quad \Delta t = t - t_0,$$

where $W(\Delta\mathbf{r}, \Delta\mathbf{w}, \Delta t|\mathbf{r}_0, \mathbf{w}_0, t_0)$ is the probability of transition for a particle passing from the volume element $(\mathbf{r}_0, \mathbf{r}_0+d\mathbf{r}_0; \mathbf{w}_0, \mathbf{w}_0+d\mathbf{w}_0)$ of the phase space where it was at the moment t_0 to the volume element $(\mathbf{r}, \mathbf{r}+d\mathbf{r}; \mathbf{w}, \mathbf{w}+d\mathbf{w})$ during the time interval Δt. As is well known, the Kolmogoroff–Chapman equation is valid for any statistical system provided (i) the continuous functions f and W exist, f tending to zero fast enough as $|\mathbf{w}|$ tends to infinity, (ii) the random phase approximation be used.

The latter approximation means that the smallest time interval which it is permissible to consider in the corresponding statistical theory must exceed sufficiently the characteristic time τ of the inner interaction in the system (Prigogine 1962). In other words, the inequality $\Delta t \gg \tau$ being accepted, equation (2·3) is valid asymptotically in any case and states in fact independence of the future from the past. To make the treatment more intelligible, let us give some examples. In kinetic theory of rarefied or dense gases the 'inner time scale' τ has the same order of magnitude as the mean time interval between subsequent collisions of the test molecule with all the other molecules and is usually much smaller than the 'outer time scale' T characterizing microprocesses of momentum or energy exchange between adjacent macroscopic volumes of the gas. It is clear that only time intervals $t \gtrsim T$ are of interest in the continuum theory of this gas. From the statistical point of view, utilization of the random phase approximation leads to the substitution of the master equation or, in particular, the Boltzmann equation for the Liouville equation. Similarly, in the theory of Brownian movement the time τ coincides with the mean time elapsing between subsequent molecular kicks experienced by a suspended particle. It is also much smaller than the outer time scale T representing in this case the typical time interval during which the velocity of the particle varies essentially. In a way, the times τ and T are analogous to the inner and outer space scales in the theory of turbulence.

Any particle in the system under examination is affected in a random way by displacement of all the neighbouring particles as well as by other perturbations of the velocity and pressure fields of the fluid. In general, the action of these perturbations on a particle in a mixture plays the same role as the action of molecular kicks on a Brownian particle. One can easily imagine the time interval between subsequent 'perturbation kicks' being very small compared with the time T of considerable change in the particle velocity or position so that the hypothesis $T \gg \tau$ seems to be well satisfied. To a certain extent it is borne out also by the experimental evidence on fluidization (see, e.g. the discussion in Buyevich's paper (1966)).

Equation (2.3) being accepted, one gets from it in a regular manner the equation

$$\frac{\partial f}{\partial t}+\frac{1}{\Delta t}\left[\frac{\partial(f\{\Delta\mathbf{r}\})}{\partial\mathbf{r}}+\frac{\partial(f\{\Delta\mathbf{w}\})}{\partial\mathbf{w}}\right]+\frac{1}{2\Delta t}\left[\left(\frac{\partial}{\partial\mathbf{r}}*\frac{\partial}{\partial\mathbf{r}}\right):\{\Delta\mathbf{r}*\Delta\mathbf{r}\}f\right.$$
$$\left.+2\left(\frac{\partial}{\partial\mathbf{r}}*\frac{\partial}{\partial\mathbf{w}}\right):\{\Delta\mathbf{w}*\Delta\mathbf{r}\}f+\left(\frac{\partial}{\partial\mathbf{w}}*\frac{\partial}{\partial\mathbf{w}}\right):\{\Delta\mathbf{w}*\Delta\mathbf{w}\}f\right]=0. \quad (2.4)$$

Here the braces { } denote averaging over the transition probability, e.g.

$$\{\Delta\mathbf{r}\}=\iint\Delta\mathbf{r}\,W(\Delta\mathbf{r}, \Delta\mathbf{w}, \Delta t|\mathbf{r}, \mathbf{w}, t_0)\,d\mathbf{r}_0 d\mathbf{w}_0$$

and we have used the notation

$$\mathbf{a} * \mathbf{b} = \|a_i b_j\|, \quad \mathbf{A}:\mathbf{B} = A_{ij} B_{ji},$$

where $\mathbf{a}, \mathbf{b}$ and $\mathbf{A}, \mathbf{B}$ are arbitrary vectors and tensors.

The expressions for the quantities within the braces in (2.4) can be found from consideration of physical properties of the system under study. First of all, it is obvious that

$$\{\Delta \mathbf{r}\} = \mathbf{w}\Delta t + O((\Delta t)^2), \quad \{\Delta \mathbf{w}\} = \mathbf{H}^* \Delta t + O((\Delta t)^2), \tag{2.5}$$

where $\mathbf{H}^*$ is the ratio of the force acting upon the test particle to its mass, m. It is relevant to emphasize that the quantity $\mathbf{H}^*$ represents in fact the true force acting on the unit mass of the particle averaged over the time interval $T \gg \tau$, the component of the latter force with frequency of the order of τ^{-1} being dropped after such an averaging. This component being set aside altogether, we draw the conclusion that all the other quantities with the braces in (2.4) are proportional to $(\Delta t)^2$ and the corresponding terms in (2.4) vanish at $\Delta t \to 0$. This component being taken into account, we can demonstrate, in the same way as in the theory of Brownian movement, that the quantity $\{\Delta \mathbf{w} * \Delta \mathbf{w}\}$ has the order of Δt, i.e.

$$\{\Delta \mathbf{w} * \Delta \mathbf{w}\} = 2\mathbf{A}\Delta t + O((\Delta t)^2), \tag{2.6}$$

where $\mathbf{A}$ is some unknown tensor describing the diffusion in the velocity space. Such a tensor was introduced in the theory of disperse systems earlier by Houghton (1966) and Levich & Myasnikov (1966).

Taking formally $\Delta t \to 0$ and making use of (2.5) and (2.6), we obtain from (2.4) the Fokker–Planck equation for suspended particles

$$\frac{\partial f}{\partial t} + \mathbf{w}\frac{\partial f}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{w}}(\mathbf{H}^* f) + \left(\frac{\partial}{\partial \mathbf{w}} * \frac{\partial}{\partial \mathbf{w}}\right) : (\mathbf{A}f) = 0. \tag{2.7}$$

We represent further the force $\mathbf{H}^*$ in (2.7) as the sum of the mean force $\langle \mathbf{H} \rangle$ and the random resistance force $-\mathbf{c}\mathbf{w}'$ depending upon the single pseudo-turbulent variable $\mathbf{w}'$ defined in accordance with (2.2)

$$\mathbf{H}^* = \langle \mathbf{H} \rangle - \mathbf{c}\mathbf{w}'. \tag{2.8}$$

Explicit expressions for the vector $\langle \mathbf{H} \rangle$ and the tensor $\mathbf{c}$, corresponding to one possible approximate expression for the total force acting upon a particle, are given in the appendix. The influence of the random component of the total force, which depends on all the other pseudo-turbulent variables, on the particle velocity distribution is believed to be defined by the last term in (2.7), as happens, for example, in the theory of Brownian movement.

This assumption, although compatible with the preceding analysis and with the utilization of the random phase approximation, is of rather heuristic character and should be refined later in the further development of the theory. It would be attractive to introduce the aforementioned component into the expression (2.8) and, thus, into (2.7) immediately. But this is hampered with the fact that in this case the kinetic equations for conventional distributions governing the pseudo-turbulent pulsations ρ', p', $\mathbf{v}'$ have also to be formulated. It does not seem to be possible because of an unknown dependence of these distributions upon multi-

particle distribution functions. Really, the velocity and the pressure of the fluid are affected by the behaviour of many particles so that the conventional distributions ought to include some functionals depending not only upon the function $f(\mathbf{w}; t, \mathbf{r})$ but also upon the above functions for two, three, etc. particles.

Taking account of (2.8) and introducing the new variable $\mathbf{w}'$ instead of $\mathbf{w}$ into (2.7), we obtain the kinetic equation in the final form (cf. the derivation of the kinetic equation for a gas)

$$\frac{Df}{Dt}+\mathbf{w}'\frac{\partial f}{\partial \mathbf{r}}+\frac{\partial}{\partial \mathbf{w}'}\left[\left(\langle \mathbf{H}\rangle-\frac{D\langle \mathbf{w}\rangle}{Dt}-\mathbf{c}\mathbf{w}'\right)f\right]-\left(\frac{\partial f}{\partial \mathbf{w}'}*\mathbf{w}'\right):\left(\frac{\partial}{\partial r}*\langle \mathbf{w}\rangle\right)$$
$$+\left(\frac{\partial}{\partial \mathbf{w}'}*\frac{\partial}{\partial \mathbf{w}'}\right):(\mathbf{A}f)=0,$$

where
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\langle w\rangle\frac{\partial}{\partial \mathbf{r}},\quad f=f(t,\mathbf{r},\mathbf{w}')=f(\mathbf{w};t,\mathbf{r}). \tag{2.9}$$

The derivation of (2.9) is quite straightforward when one transforms the derivatives involved in (2.7) by means of the obvious relations

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}-\frac{\partial\langle \mathbf{w}\rangle}{\partial t}\frac{\partial}{\partial \mathbf{w}'},\quad \frac{\partial}{\partial r_i}=\frac{\partial}{\partial r_i}-\frac{\partial\langle \mathbf{w}\rangle}{\partial r_i}\frac{\partial}{\partial \mathbf{w}'},\quad \frac{\partial}{\partial \mathbf{w}}=\frac{\partial}{\partial \mathbf{w}'}.$$

While formulating (2.9) we have in fact assumed the transition probability W to be a continuous function. One might expect that actually this function has discontinuities representing instantaneous jumps of the velocity of the test particle caused by its direct collisions with the neighbouring particles. In principle, the additional term describing the influence of those collisions and having the same meaning as, e.g. the collision term in the Boltzmann equation, can be easily introduced into (2.7) and (2.9). The kinetic equation for a disperse system containing such a term in the classical Boltzmann form has been investigated by Levich & Myasnikov (1966) and Myasnikov (1967). In this paper for the sake of simplicity we leave direct interparticle collisions out of account altogether because of very small probability of their occurrence in the system. Indeed, the approach of particles causes a sharp increase in the pressure in the liquid layer between the particles which prevents them from direct collision (see, e.g. Brenner's paper (1961); this phenomenon is well known also in the hydrodynamic theory of lubrication). Thus, we assume that particles interact mostly through the random fields of the fluid velocity and pressure so that their occasional contacts do not play an important role in momentum and energy exchange between particles. As the detailed analysis shows, this suggestion proves to be valid for a very broad class of disperse systems encountered in practice provided their concentration is not too close to that of the state of close packing.

An essential feature of the analysis proposed here is the assumption that pseudo-turbulent motion of both phases is chaotic so that the interparticle interaction has a random character. This is the case for many systems, especially for those with high concentration (e.g. for fluidized beds). Another type of interaction resulting in cluster formation can be dominant in dilute suspensions of small particles. This interaction is caused by regular (not random) motion of

particles, and accordingly, the assumption of randomness does not hold. The effect of cluster formation is not accounted for because allowance for it is connected with a number of additional complications and is unnecessary at this stage of the treatment.

We proceed now to the discussion of another important feature of the random phase approximation used. The time scale determines also the relaxation time characterizing the establishment of local equilibrium in a statistical system (Prigogine 1962). Therefore, all states of the disperse system considered here under the assumption $t \gg \tau$ must be regarded as local equilibrium ones. This gives an opportunity to describe the situation near some 'point' of the system by means of the corresponding values of the dynamic variables at this point and to consider the quantities defining various local properties of the system as certain functions of these variables. In particular, the distribution function f can be looked upon as an implicit function of t, $\mathbf{r}$ through the dynamic variables. To make this more clear, we allude to the analogy with kinetic theory of gases in which case the concept of local equilibrium reduces to that of molecular chaos and the same reasoning can be applied in a familiar way.

The quantities $\langle w_i' w_j' \rangle$ can be obtained in principle after solution of the kinetic equation (2.9) and can also be held as certain functions of the dynamic variables. We shall be interested further in other quantities of the type of $\langle \phi' \psi' \rangle$, $\langle \phi' \psi_j' \rangle$, $\langle \phi_i' \psi_j' \rangle$ where ϕ', ψ' and $\boldsymbol{\phi}', \boldsymbol{\psi}'$ are arbitrary pseudo-turbulent scalars and vectors. It is reasonable to put forward the following relationships:

$$\left.\begin{aligned} \langle \phi' \psi' \rangle = R[\phi, \psi]\theta, \quad \langle \phi' \psi_j' \rangle = R_i[\phi, \boldsymbol{\psi}]\theta_{ij}, \\ \langle \phi_i' \phi_j' \rangle = R_{ik}[\boldsymbol{\phi}, \boldsymbol{\psi}]\theta_{kj}, \quad \theta = \operatorname{tr}\boldsymbol{\theta} = \theta_{ii}, \quad \theta_{ij} = \langle w_i' w_j' \rangle \end{aligned}\right\} \tag{2.10}$$

the quantities R, R_i, R_{ik} being regarded as some functions of the dynamic variables, repeated subscripts indicating summation. These quantities as well as the tensor $\mathbf{A}$ are at this stage unknown. However, they all describe some local properties of the system and, as follows from the above considerations, must be the same functions of the dynamic variables in an arbitrary state of local equilibrium. This enables us to estimate them correctly by considering only the state of total equilibrium when all the dynamic variables do not depend upon t and $\mathbf{r}$ (apart from the mean pressure $\langle p \rangle$ which can depend on $\mathbf{r}$ linearly).

The relations (2.10) are valid, of course, when fluctuations in the fluid velocity and pressure are induced only by pseudo-turbulent pulsations of the particles. Pseudo-turbulence under study has certainly nothing in common with conventional turbulence in one-phase media, the latter being taken to be absent. That it is often relevant in practice, can be seen, for example, from Bagnold's paper (1956). In a sense pseudo-turbulence is similar to motion of a fluid filtrating through a porous body with random porosity investigated by Buyevich, Leonov & Safrai (1969). However, in the latter case the statistical characteristics of porosity are steady and given *a priori* whereas in disperse systems it is pseudo-turbulence itself that mainly determines them. Pseudo-turbulence is an intrinsic property of a disperse system, but by no means the result of hydrodynamic instability of the mean flow. If the latter occurs, additional large-scale perturbations similar to

secondary flows or turbulence in hydrodynamics of one-phase media may be important, but are not considered below.

We conclude this section by summarizing three major assumptions under which (2.9) and (2.10) are valid. First, the Kolmogoroff–Chapman equaton (2.3) with a continuous function W is supposed to be true when the smallest value of Δt considered in the theory is well above the inner time scale τ. This implies also two important consequences: (i) direct interparticle collisions are absent or play only a minor role, and (ii) any state of the disperse system under study exhibits the property of being a local equilibrium state. Second, the force $\mathbf{H}^*$ in (2.9) is represented in the form (2.8), the influence of the random pulsations $\rho', p', \mathbf{v}'$ upon the distribution function f being accounted for by the term in (2.9) which involves the diffusion in the velocity space. And finally, conventional turbulence of the fluid is absent or, at any rate, does not play an important role as compared with pseudo-turbulence.

3. Dynamic equations for the dispersed and the liquid phases

We turn now to the derivation of equations governing the mass and momentum conservation in the mean flow of both phases of the system regarded as co-existing interacting continua. These equations for the dispersed phase can be readily obtained from (2.9) in a standard manner (Chapman & Cowling 1952). Multiplying the latter equation by mf and by $m\mathbf{w}'f$ and integrating it over

$$\mathbf{w}'\,(-\infty < |w'| < \infty)$$

we get

$$\left.\begin{aligned} &\frac{D}{Dt}(d_1\langle\rho\rangle) + d_1\langle\rho\rangle\frac{\partial\langle\mathbf{w}\rangle}{\partial\mathbf{r}} = 0,\\ &d_1\langle\rho\rangle\frac{D\langle\mathbf{w}\rangle}{Dt} = -\frac{\partial\mathbf{P}^{(p)}}{\partial\mathbf{r}} + d_1\langle\rho\rangle\,\mathbf{g} + \langle\mathbf{F}^{(i)}\rangle. \end{aligned}\right\} \tag{3.1}$$

Here we have made use of the equality (see the appendix)

$$d_1\langle\rho\rangle\,\langle\mathbf{H}\rangle = d_1\langle\rho\rangle\,\mathbf{g} + \langle\mathbf{F}^{(i)}\rangle,$$

where $\mathbf{F}^{(i)}$ is the interaction force between two phases related to the unit volume of the mixture and $\mathbf{g}$ is the acceleration caused by external body force.

The tensor $\mathbf{P}^{(p)}$ defines the pseudo-turbulent stresses arising in the dispersed phase:

$$\mathbf{P}^{(p)} = d_1\langle\rho\rangle\,\langle\mathbf{w}' * \mathbf{w}'\rangle = d_1\langle\rho\rangle\,\boldsymbol{\theta}. \tag{3.2}$$

The physical origin of these stresses is just the same as that of pressure and of viscous stresses in a molecular gas. The symmetrical tensor $\mathbf{P}^{(p)}$ describes in fact the momentum transfer caused by pseudo-turbulent motion of the particles. It is worth noting also that these stresses are similar in a sense to the usual Reynolds stresses appearing in turbulent flow of one-phase fluid. The equations (3.1) describe conservation of mass and momentum of the dispersed phase looked upon as a continuum. It is important that they are independent of the unknown tensor $\mathbf{A}$ in (2.9).

It is possible to derive in the same way an equation governing the transfer of

the quantity $\boldsymbol{\theta}$ in the mean flow for which purpose one has to multiply (2.9) by $m\boldsymbol{\theta}f$ and to integrate it over $\mathbf{w}'$ once more. Then one obtains the equation

$$\frac{D}{Dt}(d_1\langle\rho\rangle\boldsymbol{\theta})+d_1\langle\rho\rangle\boldsymbol{\theta}\frac{\partial\langle\mathbf{w}\rangle}{\partial\mathbf{r}}+d_1\langle\rho\rangle S\left\{\boldsymbol{\theta}\left(\frac{\partial}{\partial\mathbf{r}}*\langle\mathbf{w}\rangle\right)+\mathbf{c}\boldsymbol{\theta}+\mathbf{A}\right\}+m\frac{\partial\mathbf{X}}{\partial\mathbf{r}}=0, \quad (3.3)$$

$$\mathbf{X}=\int(\mathbf{w}'*\mathbf{w}'*\mathbf{w}')f(t,\mathbf{r},\mathbf{w}')\,d\mathbf{w}', \quad S\{\mathbf{B}\}=\|B_{ij}+B_{ji}\|,$$

where $S\{\mathbf{B}\}$ denotes the symmetrization of an arbitrary tensor $\mathbf{B}$. The convective terms in (3.3) can be simplified by using the first equation (3.1). It follows from (3.1) and (3.3) that

$$d_1\langle\rho\rangle\frac{D\theta}{Dt}+2\mathbf{P}^{(p)}:\left(\frac{\partial}{\partial\mathbf{r}}*\langle\mathbf{w}\rangle\right)+\mathbf{c}:\mathbf{P}^{(p)}+\mathbf{P}^{(p)}:\mathbf{c}+2d_1\langle\rho\rangle A+m\frac{\partial\mathbf{Q}^{(\theta)}}{\partial\mathbf{r}}=0, \quad (3.4)$$

$$\mathbf{Q}^{(\theta)}=\|X_{ijj}\|, \quad A=\operatorname{tr}\mathbf{A}=A_{ii}.$$

This equation determines the pseudo-turbulent energy of the particles, the flux $\mathbf{Q}^{(\theta)}$ describing the part of this energy transfer due to pseudo-turbulent motion itself. One can easily see the analogy between various terms involved in (3.4) and those in the heat conduction equation. The additional terms appearing in the former equation as compared with the latter are inherent, of course, to a two-phase flow.

To derive similar equations for the fluid phase, one has to postulate equations governing the interstitial flow of the fluid filtrating through the lattice of particles arranged in disorder. These equations we present in the form (Buyevich, Leonov & Safrai 1969)

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\rho-(1-\rho)\frac{\partial\mathbf{v}}{\partial\mathbf{r}}&=0,\\ d_0(1-\rho)\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{v}&=-\frac{\partial p}{\partial\mathbf{r}}+\frac{\partial\boldsymbol{\tau}}{\partial\mathbf{r}}+d_0(1-\langle\rho\rangle)\,\mathbf{g}-\mathbf{F}^{(i)},\\ \boldsymbol{\tau}=\mu(\rho)\,\mathbf{e}, \quad \mathbf{e}&=\left\|\frac{\partial v_i}{\partial r_j}+\frac{\partial v_j}{\partial r_i}-\frac{2}{3}\frac{\partial v_k}{\partial r_k}\delta_{ij}\right\|, \quad \mu(\rho)=\mu_0 S(\rho),\end{aligned}\right\} \quad (3.5)$$

where $\mu(\rho)$ is the effective shear viscosity associated with the fluid flowing through the lattice. It should not be overlooked that, in accordance with the preceding statistical analysis, the number concentration of particles must be taken equal to its mean value $n(t,\mathbf{r})$, so that the force acting upon the fluid in unit volume of the two-phase mixture can be determined as

$$d_0(1-\sigma_0 n)\,\mathbf{g}-n\mathbf{G}^{(i)}=d_0(1-\langle\rho\rangle)\,\mathbf{g}-\mathbf{F}^{(i)},$$

where $\mathbf{G}^{(i)}$ is the interaction force per particle (see the appendix). Note that a fresh interpretation of (3.5) for a steady flow of the fluid through a cloud of particles has been recently given by Tam (1969). Equations of the same type were used for the description of the interstitial flow by Anderson & Jackson (1967). As a matter of fact, equations (3.5) represent the modified Navier–Stokes equations in which allowance is made for the volume fraction occupied by particles.

The equation for the conservation of mass of the fluid phase can be obtained by averaging the first equation (3.5). We get

$$\left(\frac{\partial}{\partial t}+\langle\mathbf{v}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\langle\rho\rangle-(1-\langle\rho\rangle)\frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}}-\frac{\partial\mathbf{q}}{\partial\mathbf{r}}=0, \qquad \mathbf{q}=-\langle\rho'\mathbf{v}'\rangle. \tag{3.6}$$

Pseudo-turbulent motion leads to occurrence of the additional mean flow of the fluid, $\mathbf{q}$, as happens, for instance, in the theory of turbulence of one-phase media. The origin of this flow is similar to that in a porous body with random porosity in which a fluid moves.

Similarly, one obtains from the second equation (3.5) the following equation of the momentum conservation in the mean flow of the liquid phase:

$$\left.\begin{aligned} d_0\frac{\partial}{\partial t}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle)+\frac{\partial}{\partial\mathbf{r}}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle*\langle\mathbf{v}\rangle) &= -d_0\frac{\partial\mathbf{q}}{\partial t}-\frac{\partial\langle p\rangle}{\partial\mathbf{r}}-\frac{\partial\mathbf{P}^{(f)}}{\partial\mathbf{r}}-\frac{\partial\mathbf{P}^{(q)}}{\partial\mathbf{r}}\\ &\quad+2\mu_0\frac{\partial\langle\mathbf{E}\rangle}{\partial\mathbf{r}}+d_0(1-\langle\rho\rangle)\,\mathbf{g}-\langle\mathbf{F}^{(i)}\rangle,\\ \mathbf{P}^{(q)}=d_0(\mathbf{q}*\langle\mathbf{v}\rangle+\langle\mathbf{v}\rangle*\mathbf{q}),\quad \mathbf{P}^{(f)}&=d_0(1-\langle\rho\rangle)\langle\mathbf{v}'*\mathbf{v}'\rangle,\\ \langle\mathbf{E}\rangle=\left(S+\frac{1}{2}\frac{d^2S}{d\langle\rho\rangle^2}\langle\rho'^2\rangle\right)\langle\mathbf{e}\rangle+\frac{dS}{d\langle\rho\rangle}\langle\rho'\mathbf{e}'\rangle,&\quad S=S(\langle\rho\rangle). \end{aligned}\right\} \tag{3.7}$$

Note that these equations are accurate to terms of the second order in the pseudo-turbulent variables. In analysis of the terms of higher order the same difficulty arises as in the problem of closure of the system of equations for the correlation functions in the theory of turbulence.

Equation (3.7) involves additional terms caused not only by the pseudo-turbulent pulsations of the fluid velocity but also by the volume flow $\mathbf{q}$ and by fluctuations in the concentration. It is helpful in some cases to transform (3.6) and (3.7) by introducing the mean value $\langle\mathbf{Q}\rangle$, and the pseudo-turbulent fluctuation $\mathbf{Q}'$, of the total flow of the fluid by means of the relations

$$\mathbf{Q}=(1-\langle\rho\rangle)\langle\mathbf{v}\rangle+\mathbf{q},\quad \mathbf{Q}'=(1-\langle\rho\rangle)\mathbf{v}'-\rho'\langle\mathbf{v}\rangle. \tag{3.8}$$

We have then from (3.6) and (3.7)

$$\partial\langle\rho\rangle/\partial t-\partial\mathbf{Q}/\partial\mathbf{r}=0,$$

$$d_0\left[\frac{\partial}{\partial t}\langle\mathbf{Q}\rangle+\frac{\partial}{\partial\mathbf{r}}(\langle\mathbf{Q}\rangle*\langle\mathbf{v}\rangle+\langle\mathbf{Q}'*\mathbf{v}'\rangle)\right]=-\frac{\partial\langle p\rangle}{\partial\mathbf{r}}+2\mu_0\frac{\partial\langle\mathbf{E}\rangle}{\partial\mathbf{r}}+d_0(1-\langle\rho\rangle)\,\mathbf{g}-\langle\mathbf{F}^{(i)}\rangle. \tag{3.9}$$

Equations (3.6) and (3.7) or (3.9) coupled with (3.1) govern the mean flow of the disperse system under study. They differ substantially from those usually postulated (as a rule, phenomenologically) owing to occurrence of the pseudo-turbulent stresses, the flow $\mathbf{q}$ and the other pseudo-turbulent quantities in them. They have to be solved together with the kinetic equation (2.9) in order to express the function $f(t,\mathbf{r},\mathbf{w}')$ in terms of the dynamic variables and of physical properties of both phases, to calculate further the tensor

$$\boldsymbol{\theta}=\langle\mathbf{w}'*\mathbf{w}'\rangle=\frac{1}{n}\int(\mathbf{w}'*\mathbf{w}')f(t,\mathbf{r},\mathbf{w}')\,d\mathbf{w}', \tag{3.10}$$

and to find all the other pseudo-turbulent quantities involved in the dynamic equations for the mean flow in accordance with (2.10).

As a result, we thus have the set of nine scalar equations (2.9), (3.1), (3.6) and (3.7) for the determination of the nine unknown functions f, $\langle\rho\rangle$, $\langle p\rangle$, $\langle v_i\rangle$ and $\langle w_i\rangle$ $(i = 1, 2, 3)$. The problem of solution of the kinetic equation for suspended particles is analogous in a sense to that encountered in the kinetic theory of gases (Chapman & Cowling 1952) and may be treated with the help of the Chapman–Enskog method in the modified form suggested by Buyevich (1970*a*). This offers an opportunity to formulate the system of successive approximations in hydromechanics of disperse systems corresponding to the well known Eulerian, Navier–Stokes and Burnett approximations in hydromechanics of gases.

There exists one essential difference between the above problem and that in kinetic theory of gases, namely, equations (2.9) and (2.10) include the unknown quantities $\mathbf{A}$ and R, R_i, R_{ij} which must also be represented as some functions of the dynamic variables. To solve the last problem some independent method of treatment of pseudo-turbulence is evidently needed. We develop such a method in § 4 making use of certain properties of local equilibrium states discussed in § 2.

4. The structure of pseudo-turbulence in an equilibrium state

The quantities $\mathbf{A}$, R, R_i and R_{ij} represent some local properties of the disperse systems and hence, as was pointed out in § 2, must be expressed in terms of the dynamic variables in any state of local equilibrium in the same manner as in a true equilibrium state in which all these variables do not depend upon t and $\mathbf{r}$ at all, except for the mean pressure $\langle p\rangle$ which can depend linearly upon $\mathbf{r}$. Therefore, to determine these quantities as functions of the dynamic variables, it is quite sufficient to consider pseudo-turbulent motion in the state of complete equilibrium. In doing so, we get simultaneously explicit expressions for various pseudo-turbulent quantities involved in (2.10) which are valid in this state of the disperse system. To indicate that those expressions are related to the equilibrium state, we denote them henceforth by a superscript zero.

First of all, we have to write stochastic equations for all the pseudo-turbulent variables. The Langevin equation for one particle has the form

$$m\,d\mathbf{w}/dt = m\mathbf{g} + \mathbf{G}^0 \quad \text{or} \quad d_1\langle\rho\rangle\,d\mathbf{w}/dt = d_1\langle\rho\rangle\,\mathbf{g} + \mathbf{F}^0, \tag{4.1}$$

where the derivation is carried out along the trajectory of the particle. The expressions for $\mathbf{G}^0$ and $\mathbf{F}^0$ are obtained from those for $\mathbf{G}^{(i)}$ and $\mathbf{F}^{(i)}$ by dropping all the derivatives of the dynamic variables with respect to t and $\mathbf{r}$ (save for the constant pressure gradient $\nabla\langle p\rangle$, of course). Subtracting the corresponding averaged equation from (4.1), one obtains the stochastic equation governing the random motion of particles

$$d_1\langle\rho\rangle\,d\mathbf{w}'/dt = \mathbf{F}^{0\prime}, \tag{4.2}$$

the force $\mathbf{F}^{0\prime}$ being dependent upon various dynamic and pseudo-turbulent variables.

Similarly, subtracting the dynamic equations (3.6) and (3.7) from the Navier–

Stokes equations (3.5) and neglecting the terms containing products of various pseudo-turbulent variables, we obtain the stochastic equations governing the random motion of the fluid

$$\left.\begin{aligned} \left(\frac{d}{dt}+\langle\mathbf{u}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\rho'-(1-\langle\rho\rangle)\frac{\partial\mathbf{v}'}{\partial\mathbf{r}} &= 0, \\ d_0(1-\langle\rho\rangle)\left(\frac{d}{dt}+\langle\mathbf{u}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{v}' &= -\frac{\partial p'}{\partial\mathbf{r}}+\mu_0 S\frac{\partial\mathbf{e}'}{\partial\mathbf{r}}-\mathbf{F}^{0\prime}, \\ \mathbf{u}=\mathbf{v}-\mathbf{w}, \quad \langle\mathbf{u}\rangle=\langle\mathbf{v}\rangle-\langle\mathbf{w}\rangle, \quad \mathbf{u}'&=\mathbf{v}'-\mathbf{w}'. \end{aligned}\right\} \tag{4.3}$$

The time derivatives here are also taken along the particle path.

Neglect of the pseudo-turbulent terms of the second order of magnitude is inherently reasonable, since, all pseudo-turbulent variables are small as compared with the corresponding dynamic variables. However, some support for this neglect can be given even in the case when the former variables are comparatively large. The terms omitted from (4.3) can be written in a general form $\gamma(\phi'\psi'-\langle\phi'\psi'\rangle^0)$, where γ depends in some way upon the dynamic variables and ϕ', ψ' are arbitrary pseudo-turbulent variables. One might assume the representative time interval, during which such quantities are changing considerably, to be roughly equal to the time scale τ so that these quantities could be looked upon as random functions of time with independent increments and, the asymptotics $t \gtrsim T \gg \tau$ being accepted, ought to vanish on averaging over the time interval of the order of T.

This idea is not a new one. The possibility of its use in hydrodynamics has been known for some time. Landau & Lifshitz (1957) have proposed the addition of artificial terms of the same kind into equations of hydromechanics of one-phase media, and Edwards (1964) and Novikov (1963) have applied this idea to the theory of turbulence.

Note that it is important to drop the above terms in our analysis in order to utilize for further investigation the efficient technique of the correlation theory of stationary random processes. Therefore, we are compelled to make now the corresponding assumption without further comment. One can believe in truth that the final results obtained will still be right in a qualitative, if not in a quantitative, way even in the case when the pseudo-turbulent variables are large. In accordance with this assumption, we have to retain those terms in the expression for the force $\mathbf{F}^{0\prime}$ which depend upon the pseudo-turbulent variables linearly (see the appendix).

We introduce further the representations of random functions through stochastic Fourier–Stieltjes integrals, viz.

$$\left.\begin{aligned} \rho' &= \iint e^{i(\omega t+\mathbf{kr})}dZ_\rho, \quad p' = \iint e^{i(\omega t+\mathbf{kr})}dZ_p, \\ \mathbf{v}' &= \iint e^{i(\omega t+\mathbf{kr})}d\mathbf{Z}_v, \quad \mathbf{w}' = \iint e^{i(\omega t+\mathbf{kr})}d\mathbf{Z}_w, \end{aligned}\right\} \tag{4.4}$$

where integration is carried out over all frequencies ω and over the entire wave-number space $\mathbf{k}$. Substituting (4.4) into (4.2) and (4.3), we obtain the set of linear

algebraic equations for the spectral measures dZ_ρ, dZ_p, $d\mathbf{Z}_v$ and $d\mathbf{Z}_w$. These equations have the form

$$\left.\begin{aligned} id_1\langle\rho\rangle\,\omega\,d\mathbf{Z}_w &= d\mathbf{Z}_F, \\ (\omega+\langle\mathbf{u}\rangle\,\mathbf{k})\,dZ_\rho-(1-\langle\rho\rangle)\,\mathbf{k}\,d\mathbf{Z}_v &= 0, \\ id_0(1-\langle\rho\rangle)\,(\omega+\langle\mathbf{u}\rangle\,\mathbf{k})\,d\mathbf{Z}_v &= -i\mathbf{k}\,dZ_p-\mu_0 S[k^2 d\mathbf{Z}_v+\tfrac{1}{3}\mathbf{k}(\mathbf{k}\,d\mathbf{Z}_v)]-d\mathbf{Z}_F. \end{aligned}\right\} \quad (4.5)$$

Here $d\mathbf{Z}_F$ is the spectral measure of the interaction force $\mathbf{F}^{0\prime}$. Using the expression for $\mathbf{F}^{(i)}$ derived in the appendix, we get

$$\begin{aligned} d\mathbf{Z}_F &= d_0\langle\rho\rangle(\beta_1 K_1+\beta_2 K_2\langle u\rangle+i\omega\eta)\,d\mathbf{Z}_u+\beta_2 K_2(\mathbf{u}_0\,d\mathbf{Z}_u)\langle\mathbf{u}\rangle \\ &\quad+\left(\beta_1\frac{dK_1}{d\langle\rho\rangle}\langle\mathbf{u}\rangle+\beta_2\frac{dK_2}{d\langle\rho\rangle}\langle u\rangle\langle\mathbf{u}\rangle+i\omega\frac{d\eta}{d\langle\rho\rangle}\langle\mathbf{u}\rangle\right)dZ_\rho-i\mathbf{k}\langle\rho\rangle\,dZ_p, \\ d\mathbf{Z}_u &= d\mathbf{Z}_v-d\mathbf{Z}_w, \quad \mathbf{u}_0=\langle\mathbf{u}\rangle/\langle u\rangle. \end{aligned} \quad (4.6)$$

The meaning of the symbols used is given in appendix.

One can readily obtain from (4.5) and (4.6) the relations between dZ_ρ and all the other spectral measures. Hence the coefficients N_p, $\mathbf{N}_v$, $\mathbf{N}_w$ in the relations

$$dZ_p=N_p(\omega,\mathbf{k},\langle\phi\rangle)\,dZ_\rho, \quad d\mathbf{Z}_v=\mathbf{N}_v(\omega,\mathbf{k},\langle\phi\rangle)\,dZ_\rho, \quad d\mathbf{Z}_w=\mathbf{N}_w(\omega,\mathbf{k},\langle\phi\rangle)\,dZ_\rho \quad (4.7)$$

can be regarded as some known functions of ω, $\mathbf{k}$ and $\langle\phi\rangle$, the latter symbol representing all the dynamic variables.

In accordance with (4.7), all the spectral measures are expressed in terms of dZ_ρ. This is not simply a matter of convenience, since the latter spectral measure can be singled out on the basis of physical considerations. Really, it is the concentration fluctuations which originate pseudo-turbulence of both phases. Besides (and it is especially important), the spectral theory of these fluctuations can be constructed in an independent way and its results can be used here (Buyevich 1970*b*).

Equations (4.7) enable us to derive expressions for pseudo-turbulent spectral densities of interest. Indeed, we have the equation for the spectral density of arbitrary scalar pseudo-turbulent variables ϕ' and ψ':

$$\begin{aligned} \Psi^0_{\phi,\psi}(\omega,\mathbf{k},\langle\phi\rangle) &= \lim_{\mathbf{dk},\,d\omega\to 0}\frac{\langle dZ^*_\phi\,dZ_\psi\rangle}{d\mathbf{k}\,d\omega} \\ &= N^*_\phi(\omega,\mathbf{k},\langle\phi\rangle)\,N_\psi(\omega,\mathbf{k},\langle\phi\rangle)\,\Psi^0_{\rho,\rho}(\omega,\mathbf{k},\langle\phi\rangle), \end{aligned} \quad (4.8)$$

so that all the spectral densities can be expressed in terms of this for the random process $\rho'(t,\mathbf{r})$. The explicit representations for the latter density has been given by Buyevich (1970*b*). The utilization of this representation leads to the final determination of the spectral densities characterizing equilibrium pseudo-turbulence. Various correlation functions can be calculated in a standard way, viz.

$$\begin{aligned} C^0_{\phi,\psi}(t',\mathbf{r}',\langle\phi\rangle) &= \langle\phi'(t,\mathbf{r})\,\psi'(t+t',\mathbf{r}+\mathbf{r}')\rangle \\ &= \iint e^{i(\omega t'+\mathbf{k}\mathbf{r}')}\Psi^0_{\phi,\psi}(\omega,\mathbf{k},\langle\phi\rangle)\,d\omega\,d\mathbf{k}. \end{aligned} \quad (4.9)$$

The averaged quantities of the type of $\langle\phi'\psi'\rangle^0$ are obtained from (4.9) at $t'=0$ and $\mathbf{r}'=0$. Having these quantities at our disposal, it is not difficult to obtain representations for all the quantities R solving for this purpose the algebraic equations (2.10) in the equilibrium state.

It is worth noting that this technique allows one to find also some other important characteristics of equilibrium pseudo-turbulence. For example, the tensor of the pseudo-turbulent diffusivities for the suspended particles can be found in a straight-forward way as

$$\mathbf{D}^0(\langle\phi\rangle) = \int_0^\infty \mathbf{C}^0_{w,w}(t', 0, \langle\phi\rangle)\, dt' = \int \mathbf{\Psi}^0_{w,w}(0, \mathbf{k}, \langle\phi\rangle)\, d\mathbf{k}, \tag{4.10}$$

where $\mathbf{\Psi}^0_{w,w}$ is the spectral tensor of the random vector process $\mathbf{w}'(t, \mathbf{r})$. This definition of $\mathbf{D}^0$ is permissible in view of the fact that the treatment of pseudo-turbulence is performed actually in the co-ordinate system connected with some moving particle.

Now we turn to the determination of the tensor $\mathbf{A}$ of the diffusion in velocity space occurring in (2.9). If one writes this equation for the equilibrium state, it would not contain any time or space derivatives. Therefore, this equation can be solved easily, its solution depending not only upon the dynamic variables and $\mathbf{w}'$ but also upon the components of the tensor $\mathbf{A}$. By definition, this tensor is symmetrical. That is why it has only six independent components which can be readily obtained from the solution of the equations

$$\theta^0_{ij} = \frac{1}{n}\int w'_i w'_j f^0(t, \mathbf{r}, \mathbf{w}')\, d\mathbf{w}', \quad \theta^0_{ij} = \langle w'_i w'_j\rangle^0, \tag{4.11}$$

whose left-hand sides are already known and right-hand sides depend upon the components of $\mathbf{A}$. Thus, the problem of the expression of the pseudo-turbulent quantities through the dynamic variables is solved in principle.

We have been concerned above only with the key aspects of the statistical theory of disperse systems and have left the explicit calculation of various characteristics of pseudo-turbulence, the solution of the kinetic equation, etc., out of account completely. We do believe, however, that this should not prevent the reader from gaining a proper understanding of the theory proposed; further details including calculation and the comparison with experiments will be presented in the subsequent parts of this paper.

Appendix

In this appendix we turn to the examination of an approximate expression for the total force acting on one particle in a concentrated disperse system. To derive that expression, we start from the known Basset form for the force $\mathbf{G}$ acting upon an isolated particle at small Reynolds number, Re,

$$\left.\begin{gathered} \mathbf{G} = m\mathbf{g} + \mathbf{G}^{(i)}, \quad \mathbf{G}^{(i)} = \mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{G}_4, \\ \mathbf{G}_1 = -\sigma_0\, \partial p/\partial \mathbf{r}, \quad \mathbf{G}_2 = \sigma_0 d_0 \beta \mathbf{u}, \quad \mathbf{G}_3 = \tfrac{1}{2}\sigma_0 d_0 d\mathbf{u}/dt, \quad \mathbf{u} = \mathbf{v} - \mathbf{w}, \\ \mathbf{G}_4 = \sigma_0 d_0 \gamma \int_{-\infty}^{t} \frac{d\mathbf{u}}{dt}\bigg|_{t=t'} \frac{dt'}{(t-t')^{\frac{1}{2}}}, \quad \beta = \frac{9\nu_0}{2a^2}, \quad \gamma = \frac{9}{2a}\left(\frac{\nu_0}{\pi}\right)^{\frac{1}{2}}. \end{gathered}\right\} \tag{A 1}$$

There is general agreement on the meaning of various terms in (A 1). Note only that $\mathbf{v}$ and p denote the fluid velocity and pressure gradient unperturbed by a single particle, and the length scale of their variation exceeds sufficiently the particle radius a.

Our problem is to generalize (A 1) to situations in which the Reynolds number is not necessarily small and there are many particles in a fluid. To this end, we utilize, as usual, the point force approximation and some experimental results. This approximation being accepted, we have the former expression for the force $\mathbf{G}_1$ resulting from the pressure field unperturbed by the given particle, i.e.

$$\mathbf{G}_1 = \oint p(\mathbf{r}' - \mathbf{r})\, dS \approx -\sigma_0\, \partial p/\partial \mathbf{r}, \tag{A 2}$$

where integration is carried out over the particle surface. For the steady drag force experienced by the particle we can use one of numerous empirical expressions obtained from experiments with a fluid flowing through a cloud or lattice of motionless particles. It is quite permissible to write such an expression in the following general form:

$$\mathbf{G}_2 \approx \sigma_0 d_0 [\beta_1 K_1(\rho) + \beta_2 K_2(\rho) u]\, \mathbf{u}. \tag{A 3}$$

In some particular cases expressions of the type of (A 3) have a good theoretical foundation (see, e.g. Tam's paper (1969)). Only the first term in (A 3) is significant at $Re \ll 1$, in the opposite case ($Re \gg 1$) only the second term is important. The functions $K_1(\rho)$ and $K_2(\rho)$ describe the influence of neighbouring particles on the local fluid flow in the vicinity of the given particle and on the drag force experienced by it. They must meet the conditions

$$K_1(0) = K_2(0) = 1, \quad dK_1/d\rho = dK_2/d\rho > 0.$$

The explicit representations for the coefficients β_1, β_2 and for the functions $K_1(\rho)$, $K_2(\rho)$ are inessential for the main purposes of this paper and are not listed here.

The force $\mathbf{G}_3$ caused by the instantaneous acceleration of the apparent mass of the adjacent fluid may be represented, by analogy with (A 1), in one of the two following forms:

$$\mathbf{G}_3 = \sigma_0 d_0 \eta(\rho)\, d\mathbf{u}/dt \quad \text{or} \quad \mathbf{G}_3 = \sigma_0 d_0 d(\eta(\rho)\, \mathbf{u})/dt, \tag{A 4}$$

where the unknown function $\eta(\rho)$ replaces the coefficient $\frac{1}{2}$ in (A 1). Unfortunately, as far as the author knows there are no empirical or theoretical representations for this function. To estimate it approximately, we consider the motion of an ideal fluid through the cloud of particles making use of the well known cell model. Thus, we assume that the velocity of the relative motion of some particle is equal to $-\mathbf{u}_*$ and, using the spherical co-ordinate system, write the boundary conditions for the flow potential ϕ in the form

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = -u_* \cos\theta, \quad \left.\frac{\partial \phi}{\partial r}\right|_{r=b} = 0, \quad b = a\rho^{-\frac{1}{3}}, \quad \cos\theta = -(\mathbf{r}\mathbf{u}_*)/(ru_*),$$

where b is the cell radius. The solution of the Laplace equation for ϕ under those boundary conditions is

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta, \quad A = \frac{\rho u_*}{1-\rho}, \quad B = \frac{\frac{1}{2}a^3 u_*}{1-\rho}.$$

The kinetic energy of the fluid inside the cell can be represented in the form

$$E = \tfrac{1}{2} d_0 \int_{a<r<b} |\nabla\phi|^2\, dr = \tfrac{1}{4}\sigma_0 d_0 u_*^2 \left[1 + \frac{\rho}{1-\rho}\left(3 + \frac{\frac{2}{3}\ln\rho}{1-\rho}\right)\right].$$

The fluid velocity equals $\mathbf{u}_* + \nabla\phi$ in the co-ordinate system connected with the moving particle. Hence the corresponding mean velocity, obtained by averaging over the cell volume occupied with the liquid, is

$$u_x = u = \frac{1}{\sigma - \sigma_0}\int_{a<r<b}\left(u_* + \frac{\partial\phi}{\partial x}\right)dr = u_*\left[1 + \frac{\rho}{1-\rho}\left(1 + \frac{\frac{1}{3}\ln\rho}{1-\rho}\right)\right],$$

$$u_y = u_z = 0.$$

This formula enables us to express the quantity E in terms of the velocity u and to obtain in a regular manner the total momentum of the perturbed flow of the fluid. As a result, we get for the force $\mathbf{G}_3$ the second expression (A 4) in which

$$\eta(\rho) \approx \frac{1}{2}\left[1 + \frac{\rho}{1-\rho}\left(3 + \frac{\frac{2}{3}\ln\rho}{1-\rho}\right)\right]\left[1 + \frac{\rho}{1-\rho}\left(1 + \frac{\frac{1}{3}\ln\rho}{1-\rho}\right)\right]^{-1}. \tag{A 5}$$

As the quantity ρ increases from zero, the function $\eta(\rho)$ decreases first from $\frac{1}{2}$, achieves the minimum and then increases ($\eta(1) = \frac{7}{4}$). Thus, $\eta(\rho)$ is of the order of magnitude of unity for all the values of ρ of interest.

The natural phenomenological generalization of the formula for $\mathbf{G}_4$ can be represented in the form

$$\mathbf{G}_4 = \sigma_0 d_0 \gamma \int_{t-t_0}^{t} \zeta(\rho)\left.\frac{d\mathbf{u}}{dt}\right|_{t=t'} \frac{dt'}{(t-t')^{\frac{1}{2}}} \tag{A 6}$$

or in the form which differs from that in (A 6) by the introduction of the unknown function $\zeta(\rho)$ under the symbol of differentiation (cf. (A 4)). Here t_0 is the characteristic time interval during which changes in the local fluid velocity within the cell continue to affect the force acting upon the particle. This quantity can be estimated as the time of propagation of perturbations induced by the particle from its surface to that of the cell. This is quite compatible with the major idea of the cell model used according to which perturbations outside any cell do not affect processes occurring inside it. Therefore, such an estimation of t_0 seems to be quite natural. The propagation velocity of perturbations has the order of magnitude of $(\nu_0\omega)^{\frac{1}{2}}$ where ω is the frequency of particle oscillations. In the case under study we mean by ω the characteristic frequency determining changes in the velocity u, so that

$$t_0 \sim \frac{b-a}{(\nu_0\omega)^{\frac{1}{2}}} = \frac{1-\rho^{\frac{1}{3}}}{\rho^{\frac{1}{3}}}\frac{a}{(\nu_0\omega)^{\frac{1}{2}}}, \quad \omega \sim \frac{u}{a}. \tag{A 7}$$

This quantity tends to infinity as $\rho \to 0$ in accordance with (A 1). If the coefficient in (A 7), depending upon ρ, is not too small (it is always so because ρ is less than 0·6 or 0·7), we have from this equation at $Re \ll 1$

$$\omega \ll \nu_0 a^{-2} \quad \text{and} \quad t_0 \ll T_\omega \sim \omega^{-1}.$$

Therefore, it follows from (A 1) and (A 6) at $\zeta(\rho) \sim 1$,

$$G_4 \sim \sigma_0 d_0 \gamma \omega u t_0^{\frac{1}{2}} \sim \sigma_0 d_0 (\nu_0/a)^{\frac{1}{4}} \omega^{\frac{3}{4}} u \ll \sigma_0 d_0 (\nu_0/a^2)\, u \sim G_2,$$

so that we can neglect the force G_4 as compared with G_2.

In the opposite case ($Re \gg 1$) we can take $t_0 \approx \infty$ in (A 6) since $t_0 \gg T_\omega$ in this case. Substituting there $u\cos\omega t$ for u, we obtain

$$G_4 \sim \sigma_0 d_0 \gamma \omega^{\frac{1}{2}} u \sim \sigma_0 d_0 (\nu_0 \omega/a^2)^{\frac{1}{2}} u \ll \sigma_0 d_0 \omega u \sim G_3,$$

so we can neglect G_4 as compared with G_3.

This offers an opportunity to obtain an approximate expression for the interaction force $\mathbf{G}^{(i)}$ valid in a general case by dropping $\mathbf{G}_4$ altogether. This is of primary importance since otherwise (2·3) could not be formulated, because that force depends upon the history of particle motion.

Thus, we have finally instead of (A 1)

$$\left.\begin{aligned} &\mathbf{G} = m\mathbf{g} + \mathbf{G}^{(i)}, \\ &\mathbf{G}^{(i)} \approx -\sigma_0 \frac{\partial p}{\partial \mathbf{r}} + \sigma_0 d_0 [\beta_1 K_1(\rho)\mathbf{u} + \beta_2 K_2(\rho) u\mathbf{u}] + \frac{d}{dt}(\eta(\rho)\mathbf{u}), \end{aligned}\right\} \tag{A 8}$$

where $\eta(\rho)$ can be approximately estimated with the help of (A 5).

Dividing (A 8) by the particle mass m, we get a formula for the force $\mathbf{H}$ per unit mass of the particle. Hence we obtain with an accuracy to the second-order terms the following expression for $\langle\mathbf{H}\rangle$:

$$\begin{aligned} \langle\mathbf{H}\rangle \approx \mathbf{g} + \chi\Big\{ &\beta_1[K_1\langle\mathbf{u}\rangle + K_1^*\langle\rho'\mathbf{u}'\rangle + \tfrac{1}{2}K_1^{**}\langle\rho'^2\rangle\langle\mathbf{u}\rangle] + \beta_2[K_2(\langle u\rangle\langle\mathbf{u}\rangle + \langle(\mathbf{u}_0\mathbf{u}')\mathbf{u}'\rangle \\ &+ \tfrac{1}{2}\mathbf{u}_0\langle u'^2\rangle - \tfrac{1}{2}\mathbf{u}_0\langle(\mathbf{u}_0\mathbf{u}')^2\rangle + K_2^*(\langle u\rangle\langle\rho'\mathbf{u}'\rangle + \langle\rho'(\mathbf{u}_0\mathbf{u}')\rangle\langle\mathbf{u}\rangle) + \tfrac{1}{2}K_2^{**}\langle\rho'^2\rangle\langle u\rangle\langle\mathbf{u}\rangle] \\ &+ \frac{D}{Dt}[\eta\langle\mathbf{u}\rangle + \eta^*\langle\rho'\mathbf{u}'\rangle + \tfrac{1}{2}\eta^{**}\langle\rho'^2\rangle\langle\mathbf{u}\rangle] + \left\langle\left(\mathbf{w}'\frac{\partial}{\partial\mathbf{r}}\right)(\eta\mathbf{u}' + \eta^*\rho'\langle\mathbf{u}\rangle)\right\rangle\Big\} - \frac{1}{d_1}\frac{\partial\langle p\rangle}{\partial\mathbf{r}}, \\ &\chi = d_0/d_1, \quad \mathbf{u}_0 = \langle\mathbf{u}\rangle/\langle u\rangle, \quad \eta(\langle\rho\rangle) = \eta, \quad K_j = K_j(\langle\rho\rangle) \quad (j = 1, 2) \end{aligned} \tag{A 9}$$

(an asterisk above denotes differentiation with respect to $\langle\rho\rangle$).

All the pseudo-turbulent quantities involved in (A 9) can be easily expressed in terms of the dynamic variables by means of relations in § 4. The corresponding expression for the random force $\mathbf{H}' = \mathbf{H} - \langle\mathbf{H}\rangle$ has the form

$$\begin{aligned} \mathbf{H}' \approx \chi\Big\{ &\beta_1(K_1\mathbf{u}' + K_1^*\rho'\langle\mathbf{u}\rangle) + \beta_2[K_2(\langle u\rangle\mathbf{u}' + (\mathbf{u}_0\mathbf{u}')\langle\mathbf{u}\rangle) + K_2^*\rho'\langle u\rangle\langle\mathbf{u}\rangle] \\ &+ \frac{d}{dt}\left(\eta\mathbf{u}' + \eta^*\rho'\langle\mathbf{u}\rangle\right) + \left(\mathbf{w}'\frac{\partial}{\partial\mathbf{r}}\right)\eta\langle\mathbf{u}\rangle\Big\} - \frac{1}{d_1}\frac{\partial p'}{\partial\mathbf{r}}. \end{aligned} \tag{A 10}$$

Hence the representation for the tensor $\mathbf{c}$ involved in the kinetic equation can be derived as

$$\mathbf{c} = \|c_{ij}\|, \quad c_{ij} \approx \chi(\beta_1 K_1 + \beta_2 K_2\langle u\rangle)\delta_{ij} + \chi\beta_2 K_2\langle\mathbf{u}\rangle\delta_{i1}\delta_{j1} - \chi\frac{\partial}{\partial r_j}(\eta\langle u_i\rangle). \tag{A 11}$$

The expression for the force $\mathbf{F}^{(i)}$ related to the unit volume of a mixture can be readily obtained from (A 9)–(A 11). We have simply the equalities

$$\mathbf{F}^{(i)} = n\mathbf{G}^{(i)} = d_1\langle\rho\rangle\mathbf{H}^{(i)} \tag{A 12}$$

and the corresponding relationships for $\langle\mathbf{F}^{(i)}\rangle$ and $\mathbf{F}^{(i)\prime}$.

Let us emphasize once more that the expression obtained for the interaction force is essentially approximate. One might, of course, argue the validity of this

expression or its accuracy in some cases as well as the mere applicability of the point force approximation for concentrated systems. The derivation of a rigorous expression being connected with the solution of a very complex many-body problem, we think that the best conceivable criterion for adequacy of the approximate expression should consist in its agreement with experiments. So far numerical results for various pseudo-turbulent quantities obtained by the author on the basis of the theory proposed here do not contradict the experimental evidence. Irrespective of this, we have to note that the theory itself is not in the least influenced by the form of the aforementioned expression. In particular, all calculations are essentially the same when any other possible expression for the interaction force is used. Therefore, the major aims of this paper formulated in § 1 are by no means affected when the particular equations (A 9)–(A 12) are utilized and there is no reason to make things more complicated at this stage.

REFERENCES

ANDERSON, T. B. & JACKSON, R. 1967 *Ind. Engng Chem. Fund.* **6**, 527.

BAGNOLD, R. A. 1956 *Phil. Trans.* A **249**, 235.

BRENNER, H. 1961 *Chem. Engng Sci.* **16**, 242.

BUYEVICH, YU. A. 1966 *Zh. Prikl. Mekh. Tekhn. Fiz.* no. 6.

BUYEVICH, YU. A. 1970*a* *Prikl. Math. Mekh.* **34**, 308.

BUYEVICH, YU. A. 1970*b* *Zh. Prikl. Mekh. Tekhn. Fiz.* no. 6.

BUYEVICH, YU. A., LEONOV, A. I. & SAFRAI, V. M. 1969 *J. Fluid Mech.* **37**, 371.

CHANDRASEKHAR, S. 1943 *Rev. Mod. Phys.* **15**, 1.

CHAPMAN, S. & COWLING, T. G. 1952 *The Mathematical Theory of Non-Uniform Gases.* Cambridge University Press.

EDWARDS, S. F. 1964 *J. Fluid Mech.* **18**, 239.

HOUGHTON, G. 1966 *Ind. Engng Chem. Fund.* **5**, 153.

LANDAU, L. D. & LIFSHITZ, E. M. 1957 *Zh. Exper. Theor. Fiz.* **32**, 618.

LEVICH, V. G. & MYASNIKOV, V. P. 1966 *Prikl. Math. Mekh.* **30**, 467.

LYPE, E. F. 1965 *Developments in Mechanics*, vol. 2. Oxford: Pergamon.

MURRAY, J. D. 1965 *J. Fluid Mech.* **21**, 465.

MYASNIKOV, V. P. 1967 *Zh. Prikl. Mekh. Tekhn. Fiz.* no. 2.

NOVIKOV, E. A. 1963 *Zh. Exper. Theor. Fiz.* **44**, 2159.

PANTON, R. 1968 *J. Fluid Mech.* **31**, 273.

PRIGOGINE, I. 1962 *Non-Equilibrium Statistical Mechanics.* London: Interscience.

SOO, S. L. 1967 *Fluid Dynamics of Multi-Phase Systems.* London: Interscience.

TAM, C. K. W. 1969 *J. Fluid Mech.* **38**, 537.